# 11 persistent homology 2 (pre-lecture)

Thursday, March 19, 2020 2:21 AM

Let $K$ be a simplicial complex.

Let $C_p = C_p(K)$ be the group of $p$-chains,

where $c = \sum a_i \sigma_i$, where $\sigma_i \in K$ are $p$-simplices, for $c \in C_p$, and $a_i \in \mathbb{Z}_2$

$\partial_p : \partial_p \to \partial_{p-1}$ is given by $\partial_p \sigma = \sum_{j=0}^{p} [u_0, \dots, \hat{u}_j, \dots, u_p]$, the boundary homomorphism.

Giving rise to the chain complex

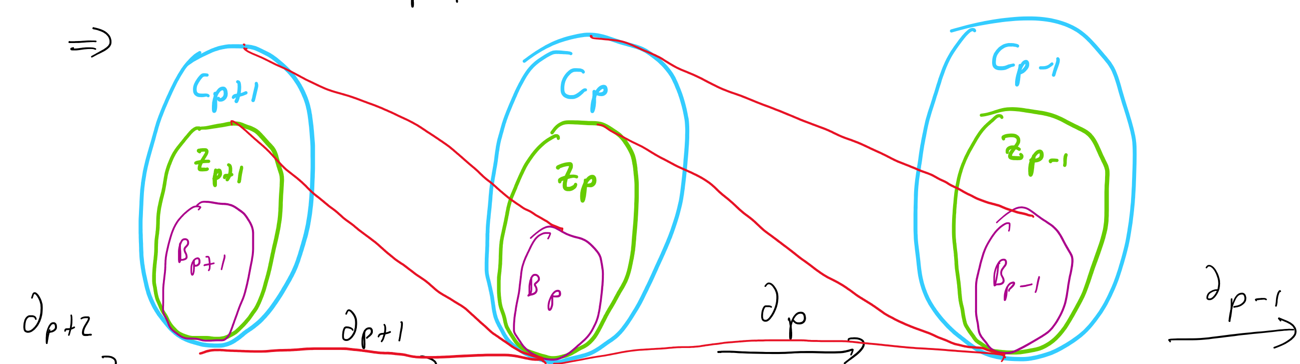
$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} C_{p-2} \to \cdots$$

Let $Z_p = Z_p(K) = \{c \in C_p \mid \partial c = 0\} = \ker \partial_p$, the subgroup of $p$-cycles.

Let $B_p = B_p(K) = \{\partial d \mid d \in C_{p+1}\} = \mathrm{Im}\, \partial_{p+1}$, the subgroup of $p$-boundaries.

**Fundamental Lemma:** $\partial_p \partial_{p+1} d = 0$.

$\Rightarrow$




Everything in $Z_{p+1}$ gets mapped to $0$, and everything in $C_{p+1}$ goes to $B_p$.

The boundary group $B_p$ is a subgroup of the cycle group $Z_p$.

**Definition:** The **pth homology group** is the pth cycle group modulo the pth boundary group, $H_p = Z_p / B_p$. The **pth Betti number** is the rank of this group, $\beta_p = \mathrm{rank}\, H_p$.

**Recall:** For $c \in Z_p$, the cosets $c + B_p$ form $H_p$.

**Definition:** The cosets of $H_p$ are referred to as a **homology class**, and any $c_1, c_2 \in c + B_p$ are **homologous**, denoted $c_1 \sim c_2$.

**Recall:** The cardinality of a group is called its order.

Recall: The cardinality of a group is called its order.

So $C_p = \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle$, where $\sigma_i \in K$ are $p$-simplices.

$\Rightarrow \operatorname{ord}(C_p) = |C_p| = 2^n$.

Note $C_p \cong \mathbb{Z}_2^n$, the group of length-$n$ bit vectors under XOR.

Recall: The rank of a vector space is its dimension, so $\operatorname{rank}(C_p) = n$.

Then $\beta_p = \operatorname{rank} H_p = \log_2 |H_p| = \log_2 \dfrac{|Z_p|}{|B_p|} = \operatorname{rank} Z_p - \operatorname{rank} B_p$.

Ex Let $K$ be a triangulation of $\mathbb{B}^k = \{x \in \mathbb{R}^k \mid |x| \leq 1\}$.

Then $H_p(K) = \{0\}$ $\forall$ $p \neq 0$, and $\beta_0 = 1$.

(hard to prove, but makes sense as there are no "holes")

Simpler: Let $K$ be the faces of a single $k$-simplex $\sigma_k$.

Claim: $H_p(K) = \{0\}$ $\forall p \neq 0$ and $\beta_0 = 1$.

proof: $H_p(K) = \{0\} \iff Z_p = B_p$.

i.e. we need to show that all $p$-cycles are $p$-boundaries for $p > 0$.

Let $\{u_0, \dots, u_k\}$ be the set of vertices.

- Note $C_k = \{0, \sigma_k\}$, $C_{k+1} = \{0\}$. $\Rightarrow B_k = \{0\}$.

  $\partial \sigma_k = \sum_{j=0}^{k} [u_0, \dots, \hat{u}_j, \dots u_k] \neq 0$ because each $k-1$-simplex appears exactly once.

  $\Rightarrow \sigma_k \notin Z_k \Rightarrow Z_k = \{0\} \Rightarrow H_p = \{0\}$.

- Let's consider $0 < p < k$.

  Let $c \in Z_p$ be a $p$-cycle with simplices of the form $[u_{i_0}, \dots, u_{i_p}]$.

Let $d$ be the set of all $p+1$-simplices of the form $[u_0, u_{i_0}, \dots, u_{i_p}]$.

Note that if $u_0$ is already in a simplex of $c$, there is no corresponding $p+1$-simplex.

We can also view $d \in C_{p+1}$ as a $p+1$-chain.

We will show $\partial d = c$.

- A $p$-simplex $\tau \in c$ that does not contain $u_0$ occurs exactly once as a face of $[u_0, u_{i_0}, \dots, u_{i_p}]$, so $\tau$ appears once in $\partial d$.
- Consider a $p$-simplex $\tau \in c$ that does contain $u_0$.

  Let $\sigma$ be the $p-1$-simplex formed by dropping $u_0$ from $\tau$.

- Consider a $p$-simplex $\tau \in c$ that does contain $u_0$.
  Let $\sigma$ be the $p-1$-simplex formed by dropping $u_0$ from $\tau$.
  We know that $\sigma$ appears an even number of times in $\partial c$, as $c \in Z_p$.
  So there must be an even number of $p$-simplices in $c$ that contain $\sigma$.
  Only one such $p$-simplex can contain both $u_0$ and $\sigma$, namely $\tau$.
  All the other such $p$-simplices have a corresponding $p+1$-simplex in $d$,
  so there are an odd number of $p+1$-simplices in $d$ that give rise to $\tau$ under the boundary operator $\partial$.
  $\Rightarrow \tau$ appears an odd number of times in $\partial d$.

- Consider a $p$-simplex $\tau \notin c$ that does contain $u_0$.
  Let $\sigma$ be the $p-1$-simplex formed by dropping $u_0$ from $\tau$.
  The same argument above shows that an even # of simplices in $c$ contain $\sigma$.
  None of those simplices can contain both $u_0$ and $\sigma$, since $\tau \notin c$.
  So all of them have a corresponding $(p+1)$-simplex in $d$.
  $\Rightarrow \tau$ appears an even number of times in $\partial d$.

- Consider a $p$-simplex $\tau \notin c$ that does not contain $u_0$.
  In order for $\tau \in \partial d$, there must exist some vertex $u'$ s.t. $[u', \tau] \in d$. But $u' \neq u_0$, because that would contradict $\tau \notin c$, based on the construction of $d$.
  And if $u' \neq u_0$, then $u_0 \notin [u', \tau] \in d$, which also contradicts the construction of $d$.

$\Rightarrow \partial d = c$.

Hence, $Z_p = B_p$ for $0 < p < k$.

- Consider now $p = 0$.
  Note that the boundary of any vertex is $0$.
  So $Z_0 = C_0$, $|Z_0| = 2^{k+1}$.
  Suppose we have a $0$-cycle $c = u_{i_1} + \dots + u_{i_\ell}$.

  If $\ell$ is even, we can pair off vertices to form $d \in C_1$ s.t. $\partial d = c$. $\Rightarrow c \in B_0$.
  If $\ell$ is odd, we cannot, so $c \notin B_0$.

Thus, $|B_0| = \dfrac{|C_0|}{2} \Rightarrow |H_0| = \dfrac{|Z_0|}{|B_0|} = 2 \Rightarrow H_0 \cong \mathbb{Z}_2$.

Thus, $|B_0| = \frac{|C_0|}{2} \Rightarrow |H_0| = \frac{|Z_0|}{|B_0|} = 2 \Rightarrow H_0 \cong \mathbb{Z}_2.$

$\Rightarrow \beta_0 = 1.$ $\boxed{\phantom{x}}$

It is possible to define a reduced homology $\tilde{\beta}_p$ so that $\tilde{\beta}_p = \beta_p$ for $p > 0$ and $\tilde{\beta}_0 = 0$, which is more convenient sometimes since we want $\tilde{\beta}_0$ to correspond to some kind of hole, not just the number of connected components

---

**Euler-Poincare Thm:** The Euler characteristic of a topological space is the alternating sum of its Betti numbers, $\chi = \sum_{p \geq 0} (-1)^p \beta_p$.

Because if $f: X \to Y$ is a homotopy equivalence, $X$ and $Y$ have isomorphic homology groups, the triangulation we use doesn't matter.

---

## Boundary matrices

Let $K$ be a simplicial complex.

Index the p-simplices $x_1, \ldots, x_{n_p}$.

Index the (p-1)-simplices $y_1, \ldots, y_{n_{p-1}}$.

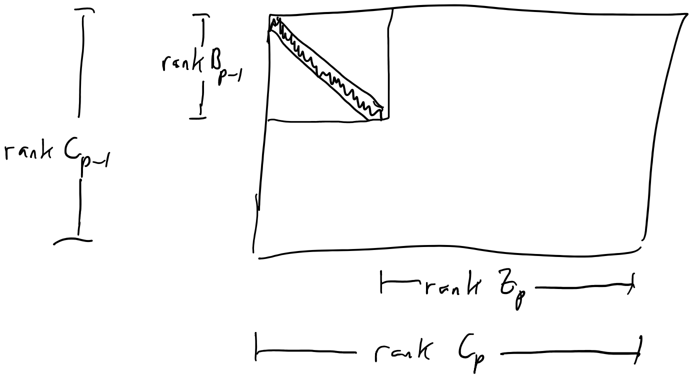
$$\partial(x_j) = \sum_{i=1}^{n_{p-1}} a_j^i y_i$$, where $a_j^i = 1$ if $y_i$ is a face of $x_j$

$a_j^i = 0$ otherwise.

Then for any p-chain $c = \sum_{j=1}^{n_p} a_j x_j$,

$$\partial_p c = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n_p} \\ a_2^1 & a_2^2 & \cdots & a_2^{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{p-1}}^1 & a_{n_{p-2}}^2 & \cdots & a_{n_{p-1}}^{n_p} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_p} \end{bmatrix},$$ when written in coordinates of $y_i$.

We can reduce this matrix via an analogue of Gaussian elimination where we can perform both row and column exchanges and sums, giving us a Smith normal form,




It turns out we can read the ranks of the boundary and cycle groups off this matrix.

Thus is why simplicial homologies are relatively easy to compute

For any data sets we can now generate a filtration of Vietoris Rips complexes by increasing radii, and then image which homologies are persistent.